Lecture 17

Thursday, March 11, 2021 4:35 PM

* Prayer
* Spiritual thought
* Answering questions ----

Change of variables

Recall : $\int_{[a,b]} f(x)\,dx$

$x = g(u)$ $\rightsquigarrow$ $\int_{[c,d]} f(g(u))\underbrace{|g'(u)|}_{\text{"stretching factor"}}du$

$[c,d]$ ↑ updated bounds

$\iint_D f(x,y)\,dA$

$x = g(u,v)$

$y = h(u,v)$

$\rightsquigarrow$ $\iint_{D'} f(g(u,v),\, h(u,v))\,\boxed{?}\,dA$

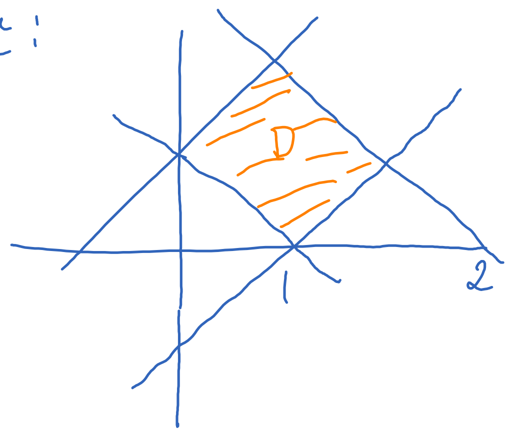
D' ↑ updated bounds

↑ stretching term

$$\text{Stretching term} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$= \left| \det \underbrace{\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{\text{Jacobian matrix}} \right|$$

Ex :



$$D = \{(x,y) : 1 \le x+y \le 2, \; -1 \le x-y \le 1 \}$$

$$\iint_D x^2 \, dA = ?$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \qquad D' = [1,2] \times [-1,1].$$

$$\rightsquigarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$$

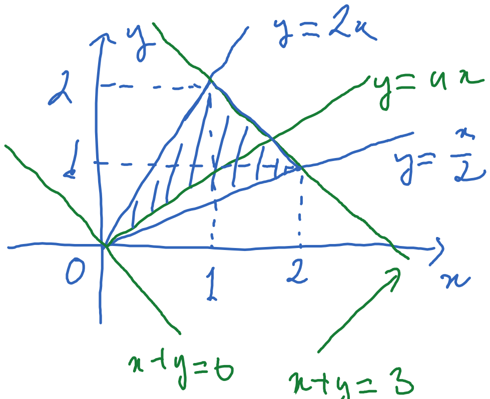
$$J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = |\det J| = \frac{1}{2}$$

$$\iint_D \cdots = \iint_{D'} \left(\frac{u+v}{2}\right)^2 \frac{1}{2} \, dA = \int_1^2 \int_{-1}^1 \frac{(u+v)^2}{2} \frac{1}{2} \, dv \, du = \cdots$$

Ex:

$$\iint_D (x+y)\, dA = \iint_{[\frac{1}{2},2]\times[0,3]} v\, dA.$$



$$\frac{1}{2} \le u = \frac{y}{x} \le 2$$

$$0 \le v = x+y \le 3$$

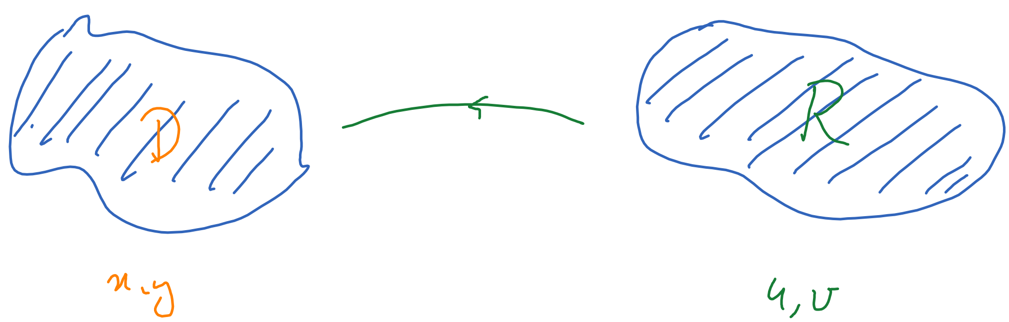
$$y = ux$$

$$v = x+y = (1+u)x$$

$$\rightsquigarrow x = \frac{v}{1+u}$$

$$y = \frac{uv}{1+u}$$

Transformation



$$x = x(u,v)$$

$$y = y(u,v)$$

$$\iint_D f(x,y)\, dA = \iint_R f(x(u,v), y(u,v))\, |\det J|\, dA.$$

Ex:

$$\int_1^2 \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$$

$u = \frac{1}{x}$

$x = \frac{1}{u}$

$dx = -\frac{1}{u^2} du$

$$\int_1^{1/2} u^2 \sin(u) \left(-\frac{1}{u^2}\right) du$$

$x = \sqrt{u}$

$dx = \frac{1}{2\sqrt{u}} du$

$$= \int_{1/2}^{1} \sin(u)\, du = \cdots$$

Ex:

$R = [0,1] \times [0,1]$

$$\begin{cases} x = u+v \\ y = u-v \end{cases} \qquad \begin{cases} x = u-v \\ y = uv \end{cases}$$

What is the region of x, y ?

Command ParametricPlot on Mathematica.

Applications

* Polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\int_D f(x,y)\,\underbrace{dA}_{dx\,dy} = \int_{D'} g(r,\theta)\,dA'$$

$x(u,v)$

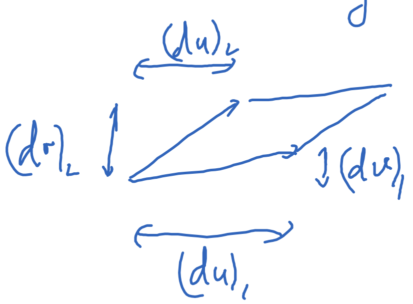
$y(u,v)$

Write double integral as $\displaystyle\iint_D f(x,y)\,\underbrace{dx\,dy}_{dA}$

$$dx\,dy = |\det J|\,du\,dv$$

$$\begin{cases} dx = x_u\,du + x_v\,dv \\ dy = y_u\,du + y_v\,dv \end{cases}$$

$$dx\,dy = x_u y_u (du)^2 + x_v y_v (dv)^2 + (x_u y_v + x_v y_u)\,du\,dv.$$



$$\langle (du)_1, (dv)_1 \rangle \times \langle (du)_2, (dv)_2 \rangle$$

$$= (du)_1 (dv)_2 - (du)_2 (dv)_1$$

$$\left.\begin{aligned} dx &= x_u (du)_1 + x_v (dv)_1 \\ 0 &= y_u (du)_1 + y_v (dv)_1 \end{aligned}\right\} \to (du)_1 = \frac{\begin{vmatrix} dx & x_v \\ 0 & x_v \end{vmatrix}}{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}} = \frac{-x_v}{|\cdots|} dx$$

$$(dv)_1 = \frac{-y_u x_v}{y_v |\cdots|} dx$$

$$\left.\begin{aligned} 0 &= x_u (du)_2 + x_v (dv)_2 \\ dy &= y_u (du)_2 + y_v (dv)_2 \end{aligned}\right\} \to (du)_2 = \frac{\begin{vmatrix} 0 & x_v \\ dy & y_v \end{vmatrix}}{|\cdots|} = \frac{-x_v}{|\cdots|} dy$$

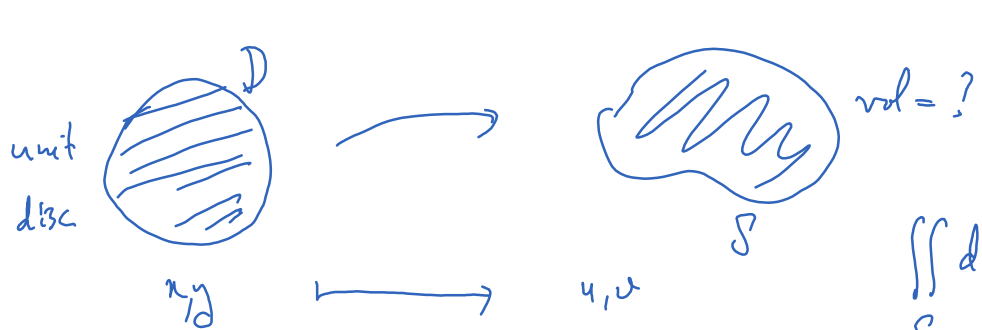
$$(dv)_2 = \frac{x_u}{|\cdots|} dy$$

$$x \qquad - \frac{x_v^2 y_u}{y_v}$$

The inverse law:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}.$$

Ex:



$u = x^2$

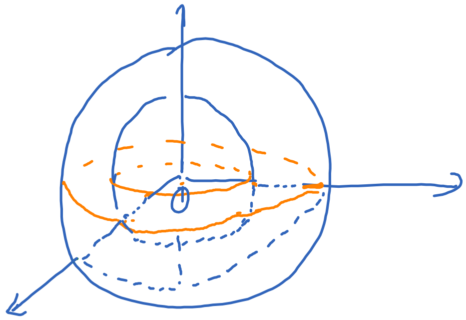
$v = xy$

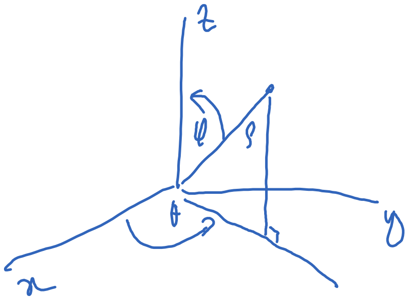
$$\iint_S du\, dv = \iint_D \underset{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}{?}\, dx\, dy$$

$$\begin{vmatrix} 2x & 0 \\ y & x \end{vmatrix} = 2x^2$$

Spherical coords

$$\iiint_E x^2 \, dV$$





$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$(x, y) = \underbrace{r(u, v)}_{\text{vector function}}$$

$$\underbrace{r(u+du, v) - r(u, v)} = r_u \, du$$